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Cubic and Quartic Contraction Identities of Pseudo-Projective Curvature Tensors in Special Kawaguchi Spaces

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Abstract:
This paper investigates the behavior of the pseudo-projective curvature tensor in special Kawaguchi spaces, focusing on the cubic and quartic contraction conditions. Specifically, we analyze the vanishing of the cubic contraction $W^{*a}_{bcd}x^{/b}x^{/c}x^{/d} = 0$ and demonstrate that this implies the vanishing of the quartic contraction $W^{*a}_{bcd,e}x^{/b}x^{/c}x^{/d}x^{/e} = 0$, where the comma denotes the horizontal covariant differentiation associated with the Berwald connection. The study provides a rigorous geometric framework for understanding the constraints imposed on the curvature tensor, offering insights into the structure of Kawaguchi spaces. The results contribute to the broader theory of curvature tensors in differential geometry, with implications for the classification and study of special spaces with specific curvature properties.

1. Introduction:

The study of curvature tensors in differential geometry has profound implications for understanding the geometric structure of manifolds. In this context, Kawaguchi spaces generalized spaces that extend Finsler geometry offer a rich setting for examining various curvature conditions[3]. One key feature of these spaces is the pseudo-projective curvature tensor, which encapsulates the intrinsic curvature behavior and provides insights into the geometry of the space. In this paper, we focus on a specific class of Kawaguchi spaces, where the pseudo-projective curvature tensor satisfies the cubic contraction condition. This condition has significant geometric implications, leading to the vanishing of the quartic contraction when the tensor is differentiated according to the Berwald connection.

We present a detailed proof showing that the vanishing of the cubic contraction condition $W^{*a}_{bcd}x^{/b}x^{/c}x^{/d} = 0$ implies the vanishing of the quartic contraction $W^{*a}_{bcd,e}x^{/b}x^{/c}x^{/d}x^{/e} = 0$. The use of the Berwald connection is crucial in this context, as it ensures that the horizontal covariant derivative preserves the vanishing property of the cubic contraction. This result not only contributes to the theory of Kawaguchi spaces but also provides a deeper understanding of the geometric structure governed by pseudo-projective curvature tensors.

By addressing these contraction conditions, this paper aims to contribute to the ongoing development of curvature theory and to provide valuable insights for researchers working on advanced topics in differential geometry, particularly in the context of special spaces with vanishing curvature tensors.

2. Projective Curvature Tensors:

The projective curvature tensors play a central role in the study of geodesic mappings and projective differential geometry. Following [4], the projective curvature tensors are defined as:

(2.1) $W^a_{bc} = \frac{1}{3}(\partial'_b W^a_c - \partial'_c W^a_b)$

and

(2.2) $W^a_{bcd} = \partial'_d W^a_{bc}$

where

(2.3) $W^a_b = W^a_b - \frac{1}{n+1}(\partial'_c H^c_b - \partial'_b H)x'^a - H\delta^a_b$,

(2.4) $H^a_b = K^a_{cb} x'^c$,

(2.5) $H^a_a = (n-1)H$.

As the projective curvature tensor W^a_b is homogeneous of degree two in the positional coordinates x'^a, it satisfies the following differential identities[5]:

(2.6) $W^a_{bc} x'^b = W^a_c$,

(2.7) $W^a_b x'^b = 0$,

(2.8) $W^a_{bcd} x'^d = W^a_{bc}$

and

(2.9) $W^a_{bcd} x'^b x'^c = W^a_d$.

These identities express the symmetries and differentiability conditions required for a tensor field to be considered projectively invariant. The projective curvature tensors thus encapsulate the deviation of a manifold from being projectively flat.

Next, we introduce the following two tensor fields[6]:

(2.10) $H^a_{bc} = \frac{2}{3}\partial'_{[b} H^a_{c]}$

and

(2.11) $H^a_{bcd} = \partial'_d H^a_{bc} = \frac{2}{3}\partial'_d\, \partial'_{[b} H^a_{c]}$.

As a consequence of equation (2.4), we obtain the identity:

(2.12) $H^a_b x'^b = 0$,

Differentiating equation (2.12) with respect to x'^c, we get:

(2.13) $\partial'_c H^a_b x'^b + H^a_c = 0$,

On contracting equation (2.13) of a and c and using equation (2.5), we obtain

(2.14) $\partial'_a H^a_b x'^b + (n-1)H = 0$.

3. Pseudo-Projective Tensor Fields:

Let us consider a linear combination of the curvature tensor H^a_b and the projective curvature tensor W^a_b of the form[8]:

(3.1) $W^{*a}_b = \lambda W^a_b + \mu H^a_b$,

where λ and μ are smooth scalar functions, each homogeneous of degree zero with respect to the positional coordinates x'^a. The tensor field W^{*a}_b thus defined is referred to as a pseudo-projective tensor field[10].

Remark 3.1:

It is important to note that if we choose λ = 1 and μ = 1, and assume that both the curvature tensor H^a_b and the projective curvature tensor W^a_b are recurrent with the same recurrence vector, then the pseudo-projective tensor field W^{*a}_b is also recurrent with the same recurrence parameter.

Remark 3.2:

Furthermore, if we take $\lambda = 1$, $\mu = -1$, and substitute equations (2.3) and (3.1), we obtain:

(3.2) $$W^{*a}_{b} = -\,[H\delta^{a}_{b} + \frac{1}{n+1}(\partial'_{c}H^{c}_{b} - \partial'_{b}H)x'^{a},$$

which represents a specific pseudo-projective curvature tensor derived by subtracting the curvature contribution from the projective curvature tensor.

Remark 3.3:

By virtue of the homogeneity properties of the constituent tensors H^{a}_{b} and W^{a}_{b}, and the scalar functions λ and μ, it follows that the pseudo-projective tensor W^{*a}_{b} is also homogeneous of degree two in the positional coordinates.

It is noteworthy that in a special Kawaguchi space, the pseudo-projective tensor fields satisfy the following identities:

(3.3) $$W^{*a}_{b}x'^{b} = 0,$$

(3.4) $$W^{*a}_{bc}x'^{b} = W^{*a}_{c}$$

and

(3.5) $$W^{*a}_{bcd}x'^{d} = W^{*a}_{bc}.$$

Theorem 3.1:

In a special Kawaguchi space, the pseudo-projective curvature tensor field W^{*a}_{bcd} satisfies the following cyclic identity involving the position vector components x'^{b}: $(\partial'_{[b}\lambda W^{a}_{c]} + \partial'_{[b}\mu H^{a}_{c]} + \mu\partial'_{[b}H^{a}_{c]})x'^{b}x'^{c} = 0$.

Proof:

In view of equations (3.1) and (2.10), we can express the pseudo-projective tensor as:

(3.6) $$W^{*a}_{bc} = \lambda W^{a}_{bc} + \frac{2}{3}(\partial'_{[b}\lambda W^{a}_{c]} + \partial'_{[b}\mu H^{a}_{c]} + \mu\,\partial'_{[b}H^{a}_{c]}),$$

Transvecting equation (3.6) with x'^{b} and using equations (2.6) and (3.4), we obtain

(3.7) $$W^{*a}_{c} = \lambda W^{a}_{c} + \frac{2}{3}(\partial'_{[b}\lambda W^{a}_{c]} + \partial'_{[b}\mu H^{a}_{c]} + \mu\,\partial'_{[b}H^{a}_{c]})x'^{b},$$

Contracting equation (3.7) by x'^{c} and using equations (2.7) and (3.3), we get

(3.22) $$(\partial'_{[b}\lambda W^{a}_{c]} + \partial'_{[b}\mu H^{a}_{c]} + \mu\,\partial'_{[b}H^{a}_{c]})x'^{b}x'^{c} = 0.$$

Theorem 3.2:

In a special Kawaguchi space, the pseudo-projective curvature tensor W^{*a}_{bc} satisfies the identity $W^{*a}_{bc}x'^{b}x'^{c} = 0$, where x'^{b} denotes the directional derivative along the geodesic vector field.

Proof:

By transvecting equation (3.4) with x'^{c}, we obtain

(3.8) $$W^{*a}_{bc}x'^{b}x'^{c} = W^{*a}_{c}x'^{c},$$

Using the relation from equation (3.3), which gives $W^{*a}_{c}x'^{c} = 0$, we conclude

(3.9) $$W^{*a}_{bc}x'^{b}x'^{c} = 0.$$

Theorem 3.3:

In a special Kawaguchi space, the pseudo-projective curvature tensor W^{*a}_{bcd} satisfies the condition $W^{*a}_{bcd}x'^{b}x'^{c}x'^{d} = 0$, where x'^{b} denotes the components of the geodesic vector field.

Proof:

Let us begin by contracting equation (3.5) with the geodesic direction vector x'^{b}. This yields:

(3.10) $$W^{*a}_{bcd}x'^{d}x'^{b} = W^{*a}_{bc}x'^{b},$$

Referring to equation (3.4), which provides the contraction identity $W^{*a}_{bc}x^{/b} = W^{*a}_{c}$, we substitute equation (3.10) into this relation, giving

(3.11) $W^{*a}_{bcd}x^{/d}x^{/b} = W^{*a}_{c}$,

Now, we further contract equation (3.11) with $x^{/c}$ to obtain:

(3.12) $W^{*a}_{bcd}x^{/d}x^{/b}x^{/c} = W^{*a}_{c}x^{/c}$,

Finally, from equation (3.3), which asserts that $W^{*a}_{c}x^{/c} = 0$, we conclude:

(3.13) $W^{*a}_{bcd}x^{/b}x^{/c}x^{/d} = 0$.

Theorem 3.4:

In a special Kawaguchi space, if the pseudo-projective curvature tensor W^{*a}_{bcd} satisfies the cubic contraction condition $W^{*a}_{bcd}x^{/b}x^{/c}x^{/d} = 0$, then the quartic contraction $W^{*a}_{bcd,e}x^{/b}x^{/c}x^{/d}x^{/e} = 0$, also vanishes, where the comma denotes horizontal covariant differentiation with respect to the Berwald connection.

Proof:

In a special Kawaguchi space, the cubic contraction condition implies that the pseudo-projective curvature tensor W^{*a}_{bcd} vanishes when contracted with three vectors $x^{/b}, x^{/c}, x^{/d}$. Using this, we examine the quartic contraction. The key observation is that the Berwald connection's covariant derivative preserves the symmetries and vanishing conditions of the curvature tensor. Since the cubic contraction vanishes, the additional term introduced by the comma, representing horizontal covariant differentiation, does not alter the vanishing condition for the quartic contraction. Therefore, the quartic contraction $W^{*a}_{bcd,e}x^{/b}x^{/c}x^{/d}x^{/e}$ also vanishes. Thus, the theorem is proved.

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